Lecture 7

6.8 - Indeterminate Forms and L'Hôpital's Rule

In Calc I, you computed the limit

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

using a complicated method. This was necessary because

$$\text{“}\frac{\sin 0}{0} = \frac{0}{0}\text{”} \ldots$$

This is called an indeterminate form of the type $\frac{0}{0}$.

Other examples of this are:

$$\lim_{x\to\infty} \frac{e^{-x}}{x^{-3}} \quad \text{and} \quad \lim_{x\to 2} \frac{\cos\left(\frac{\pi}{x}\right)}{x-2}$$

An indeterminate form of type $\frac{0}{0}$ is more formally the limit of a quotient $\frac{f(x)}{g(x)}$ where both $f(x), g(x) \to 0$.

Likewise, we can define an indeterminate form of type $\frac{\infty}{\infty}$ as the limit of a quotient $\frac{f(x)}{g(x)}$ where $f(x) \to \pm\infty$ and $g(x) \to \pm\infty$.

Examples of this type:

$$\lim_{x\to\infty} \frac{x^2}{e^x}, \quad \lim_{x\to -\frac{\pi}{2}^+} \frac{\tan x}{\ln\left(x+\frac{\pi}{2}\right)}, \quad \lim_{x\to 0^-} \frac{x^{-2}}{\csc x}$$

L'Hôpital's Rule

Let $\lim$ stand for any of

$$\lim_{x\to a}, \quad \lim_{x\to a^+}, \quad \lim_{x\to a^-}, \quad \lim_{x\to\infty}, \quad \text{or} \quad \lim_{x\to -\infty}$$

and suppose $\lim \frac{f(x)}{g(x)}$ is an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Then, if $\lim \frac{f'(x)}{g'(x)}$ is a finite number L, or $\pm\infty$, then

$$\boxed{\lim \frac{f(x)}{g(x)} \overset{H}{=} \lim \frac{f'(x)}{g'(x)}}$$

(Write $\overset{H}{=}$ when you apply L'Hôpital's rule)

(Technical assumptions: Need to assume that both $f(x)$ and $g(x)$ are differentiable in some interval around a/∞, except possibly at a, and that $g'(x)\neq 0$ in that interval).

Ex: $$\lim_{x\to 0} \frac{\sin x}{x} \overset{H}{=} \lim_{x\to 0} \frac{\cos x}{1} = \frac{1}{1} = \boxed{1}$$

of type $\frac{0}{0}$

$\lim f(x)g(x)$ is an <u>indeterminant form of type $0\cdot\infty$</u> if $\lim f(x) = 0$ & $\lim g(x) = \infty$

An example of this is $\lim\limits_{x\to\infty} x^3 e^{-x^2}$

The method of dealing with this type is to turn it into either $\frac{0}{0}$ or $\frac{\infty}{\infty}$. To turn it into:

$\frac{0}{0}$: write $f(x)g(x) = \dfrac{f(x)}{\left(\frac{1}{g(x)}\right)}$

$\frac{\infty}{\infty}$: write $f(x)g(x) = \dfrac{g(x)}{\left(\frac{1}{f(x)}\right)}$

<u>Ex</u>: $\lim\limits_{x\to\infty} \underset{\substack{\downarrow \\ \infty}}{x^3}\, \underset{\substack{\downarrow \\ 0}}{e^{-x^2}} = \lim\limits_{x\to\infty} \dfrac{x^3}{e^{x^2}} \overset{H}{=} \lim\limits_{x\to\infty} \dfrac{3x^2}{2xe^{x^2}} = \lim\limits_{x\to\infty} \dfrac{3x}{2e^{x^2}}$

$$\overset{H}{=} \lim_{x\to\infty} \frac{3}{4xe^{x^2}} = 0$$

Other common types of indeterminant forms are of the <u>type 0^0, ∞^0, 1^∞</u>: These involve limits of the form $\lim [f(x)]^{g(x)}$ where:

<u>0^0</u>: $f(x)\to 0$, $g(x)\to 0$	<u>∞^0</u>: $f(x)\to\infty$, $g(x)\to 0$	<u>1^∞</u>: $f(x)\to 1$, $g(x)\to\infty$

The method to deal with these is the same in each case:

Since e^x is continuous, and $f(x)^{g(x)} = e^{g(x)\ln[f(x)]}$, to compute $\lim f(x)^{g(x)}$:

1) Compute $\lim\left(g(x)\ln[f(x)]\right)$
2) Call this limit α. α may be finite, or $\pm\infty$
3) Then $\lim f(x)^{g(x)} = \lim\limits_{t\to\alpha} e^t$

<u>Ex</u>: $\lim\limits_{x\to 0^+} x^{\sqrt{x}}$

$$x^{\sqrt{x}} = e^{\sqrt{x}\ln x}$$

$$\lim_{x\to 0^+} \underset{\substack{\downarrow\\ 0}}{\sqrt{x}}\;\underset{\substack{\downarrow\\ -\infty}}{\ln x} = \lim_{x\to 0^+} \frac{\ln x}{x^{-1/2}} \overset{H}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{2}x^{-3/2}} = \lim_{x\to 0^+} \frac{-2}{x^{-1/2}}$$

$$= \lim_{x\to 0^+} -2\sqrt{x} = 0$$

So, $\lim\limits_{x\to 0^+} x^{\sqrt{x}} = \lim\limits_{t\to 0} e^t = \boxed{1}$

The last indeterminant form is of <u>type $\infty - \infty$</u>: this occurs with limits of the form $\lim(f(x)-g(x))$ where $f(x), g(x) \to \infty$ or $f(x), g(x) \to -\infty$.

The way to deal with these are to try to turn them into quotients, and then apply the previous methods.

<u>Ex</u>: $\lim\limits_{x\to 0^+}\Big(\underbrace{\frac{1}{x}}_{\infty} - \underbrace{\frac{1}{e^x-1}}_{\infty}\Big) = \lim\limits_{x\to 0^+}\left(\frac{e^x-1-x}{xe^x-x}\right)$

$\stackrel{H}{=} \lim\limits_{x\to 0^+}\left(\frac{e^x-1}{xe^x+e^x-1}\right) \stackrel{H}{=} \lim\limits_{x\to 0^+}\left(\frac{e^x}{e^x+xe^x+e^x}\right) = \frac{1}{1+0+1} = \boxed{\frac{1}{2}}$

(still $\frac{0}{0}$)

<u>Ex</u>: $\lim\limits_{x\to 0}(\underbrace{\csc x}_{\infty} - \underbrace{\cot x}_{\infty}) = \lim\limits_{x\to 0}\left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right)$

$= \lim\limits_{x\to 0}\left(\frac{1-\cos x}{\sin x}\right) \stackrel{H}{=} \lim\limits_{x\to 0}\left(\frac{\sin x}{\cos x}\right) = \frac{0}{1} = \boxed{0}$